

THE FREDHOLM METHOD IN POTENTIAL THEORY⁽¹⁾

BY
JOSEF KRÁL

Introduction. Let G be an open set with a compact boundary B in R^m , the Euclidean m -space. If h is a harmonic function in G such that

$$(1) \quad \int_P |\text{grad } h(x)| \, dx < \infty$$

for every bounded open set $P \subset G$, one may form the distribution Nh over the space D of all infinitely differentiable functions ψ with compact support in R^m defining

$$\langle \psi, Nh \rangle = \int_G \text{grad } \psi(x) \cdot \text{grad } h(x) \, dx.$$

This distribution will be termed the generalized normal derivative of h (compare [CC], [M], [Y]). It is easily seen that Nh has support in B . In general, Nh need not be a measure in the sense usual in distribution theory [S]. §1 of the present paper deals with generalized normal derivatives of Newtonian potentials. We denote by $C^*(B)$ the Banach space of all finite signed Borel measures with support in B ; total variation is taken as a norm in $C^*(B)$. With every $\mu \in C^*(B)$ we associate the corresponding Newtonian potential

$$U\mu(x) = \int_{R^m} p(x-y) \, d\mu(y),$$

where $p(z) = |z|^{2-m}/m-2$ or $p(z) = \log(1/|z|)$ according as $m > 2$ or $m = 2$, and we ask what necessary and sufficient condition is to be imposed on G in order that $NU\mu$ be a measure for every $\mu \in C^*(B)$. For this purpose it is useful to introduce the concept of a hit of a half-line $\{y + t\theta : t > 0\}$ on G (cf. Definition 1.5). If $n(\theta, y)$ denotes the number of such hits, then $n(\theta, y)$ is a Baire function of the variable θ on $\Gamma = R^m \cap \{\theta : |\theta| = 1\}$ and the above mentioned condition reads as follows:

$$(2) \quad \sup_{y \in B} \int_{\Gamma} n(\theta, y) \, dH_{m-1}(\theta) < \infty,$$

where H_{m-1} stands for the $(m-1)$ -dimensional Hausdorff measure.

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If G fulfills (2), then the operator

$$NU: \mu \rightarrow NU\mu$$

is bounded on $C^*(B)$ and has the form $\frac{1}{2}AI + \bar{W}^*$, where $A = H_{m-1}(\Gamma)$, I is the identity operator and $\bar{W}^*$ is adjoint to an operator $\bar{W}$ acting on the space $C(B)$ of all continuous functions on B . Some properties of $\bar{W}$, which is connected with the classical double-layer potential, are investigated in §§2–3. In particular, in §3 we show that, in case B has no isolated points, the Fredholm radius of $\bar{W}$ is the reciprocal of the quantity

$$V_0 = \lim_{r \downarrow 0} \sup_{y \in B} \left[A|d(y) - \frac{1}{2}| + \int_{\Gamma} n_r(\theta, y) dH_{m-1}(\theta) \right],$$

where $d(y)$ denotes the m -density of G at y and $n_r(\theta, y)$ is the number of hits of $\{y + t\theta : 0 < t < r\}$ on G . Relations between V_0 and the geometric structure of B are also investigated in §3. In case V_0 is sufficiently small, these results apply to the Neumann problem where the boundary condition is given by an arbitrary measure $\nu \in C^*(B)$, as treated in §4. By duality based on the Fredholm theory one obtains, as a by-product, representation of solutions of the Dirichlet problem by means of double-layer potentials.

Methods and concepts employed here are those of geometric measure theory; they have their origin in investigations connected with the Gauss-Green theorem, sets with finite perimeter and functions whose partial derivatives are measures [DG], [F], [FL], [FY], [KR], [MA], [P].

1. Normal derivatives of potentials.

1.1. *Terminology and notation.* The symbols R^m , $C^*(B)$, p , $U\mu$, D will have the meaning described in the introduction. For $M \subset R^m$ we shall denote by $\text{cl } M$, $\text{int } M$, $\text{fr } M$ and $\text{diam } M$ the closure, interior, boundary and diameter of M , respectively. H_k will stand for the k -dimensional Hausdorff measure; H_m coincides with the Lebesgue measure in R^m . We put $\Omega_r(y) = R^m \cap \{z : |z - y| < r\}$, $\Omega = \Omega_1(0)$, $\Gamma_r(y) = \text{fr } \Omega_r(y)$, $\Gamma = \Gamma_1(0)$, $A = H_{m-1}(\Gamma)$. Throughout this paragraph $G \subset R^m$ ($m \geq 2$) will be a fixed set with a compact boundary B . We shall tacitly assume that G is open. On several places, however, it will be useful to allow G to be a Borel set; this will be always pointed out explicitly.

The generalized normal derivative of a harmonic function h (satisfying (1) for every bounded open $P \subset G$) is defined as in the introduction; we shall write $N^G h$ instead of Nh if it is necessary to specify G . The reason for the terminology is obvious: if G has a smooth boundary with exterior normal n and h is smooth up to B , then

$$\langle \psi, Nh \rangle = \int_B \psi(\partial h / \partial n) dH_{m-1}.$$

If $\text{spt } \psi$ (=the support of ψ) does not meet B , then there is an open set Q with a smooth boundary such that $\text{spt } \psi \cap G \subset Q$, $\text{cl } Q \subset G$, so that

$$\langle \psi, N^G h \rangle = \langle \psi, N^Q h \rangle = 0.$$

In particular, if $N^G h$ is a (Borel) measure ν , which means that

$$\langle \psi, N^G h \rangle = \int_{R^m} \psi \, d\nu$$

for every $\psi \in D$, then $\nu \in C^*(B)$.

Variation of a (signed) measure μ on a Borel set M will be denoted by $|\mu|(M)$; for $\mu \in C^*(B)$, $|\mu|(B) = \|\mu\|$ is the norm of μ .

Simple calculation shows that, for $\mu \in C^*(B)$ and $x \in G$,

$$|\text{grad } U\mu(x)| \leq \int_B |x-y|^{1-m} d|\mu|(y),$$

whence we obtain for any bounded Borel $P \subset G$

$$(1.1) \quad \int_P |\text{grad } U\mu(x)| \, dx \leq A \text{diam}(B \cup P) \|\mu\|.$$

We see that $NU\mu$ is meaningful for every $\mu \in C^*(B)$. Our main objective in §1 is to answer the following question:

1.2. *What necessary and sufficient restrictions are to be imposed on G in order that $NU\mu$ be a measure for every $\mu \in C^*(B)$?*

1.3. REMARK. Let us agree to denote by δ_y the Dirac measure concentrated at $y \in R^m$. We have for any $\psi \in D$ and any $y \in B$

$$\langle \psi, NU\delta_y \rangle = \int_G \text{grad } \psi(x) \cdot \frac{y-x}{|y-x|^m} \, dx.$$

Direct calculation shows that, in case $Q = R^m - \{y\}$, $N^Q U\delta_y = A\delta_y$.

Let us also observe that, for $\psi \in D$ and $\mu \in C^*(B)$,

$$(1.2) \quad \langle \psi, NU\mu \rangle = \int_B \langle \psi, NU\delta_y \rangle \, d\mu(y).$$

Indeed, if $P = G \cap \text{spt } \psi$ and $K = \sup |\text{grad } \psi|$, then

$$\iint_{G \times B} \left| \text{grad } \psi(x) \cdot \frac{y-x}{|y-x|^m} \right| \, dx \, d|\mu|(y) \leq KA \text{diam}(P \cup B) \|\mu\|,$$

so that Fubini's theorem applies to

$$(1.3) \quad \iint_{G \times B} \text{grad } \psi(x) \cdot (y-x) |y-x|^{-m} \, dx \, d\mu(y);$$

it remains to notice that the two repeated integrals derived from (1.3) occur in (1.2).

Before investigating the problem 1.2 we shall answer the following simpler question:

1.4. Fix $y \in B$. What must be the shape of G in order that $NU\delta_y$ be a measure?

Let us first introduce a concept which will be useful later.

1.5. DEFINITION. If $M \subset R^k$ is a Borel set and $S \subset R^k$ is an open segment or half-line then $z \in S$ will be termed a hit of S on M provided both $S \cap M \cap \Omega_r(z)$ and $(S - M) \cap \Omega_r(z)$ have a positive linear measure for every $r > 0$.

An answer to 1.4 is included in the following proposition, which will be needed later.

1.6. PROPOSITION. Suppose that G is a Borel set. Fix $y \in R^m$, $r > 0$ and put

$$E_r(y) = D \cap \{\psi : \text{spt } \psi \subset \Omega_r(y), |\psi| \leq 1\},$$

$$D_r(y) = E_r(y) \cap \{\psi : y \notin \text{spt } \psi\}.$$

If $n_r(\theta, y)$ denotes the number (possibly 0 or ∞) of all hits of $\{y + \rho\theta : 0 < \rho < r\}$ on G , then $n_r(\theta, y)$ is a Baire function of the variable θ on Γ , the integral

$$v_r(y) = \int_{\Gamma} n_r(\theta, y) dH_{m-1}(\theta)$$

is equal to

$$\sup \left\{ \int_G \text{grad } \psi(x) \cdot \frac{y-x}{|y-x|^m} dx : \psi \in D_r(y) \right\}$$

and

$$\sup \left\{ \int_G \text{grad } \psi(x) \cdot \frac{y-x}{|y-x|^m} dx : \psi \in E_r(y) \right\} \leq A + v_r(y).$$

If $y \in B$ and G is open, then $NU\delta_y$ is a measure if and only if $v_{\infty}(y) < \infty$.

1.7. REMARK. If it is necessary to specify the set G , we write $n_r^G(\theta, y)$ and $v_r^G(y)$ instead of $n_r(\theta, y)$ and $v_r(y)$.

We postpone the proof of Proposition 1.6 to 1.11. First we establish two lemmas.

1.8. Notation. If f is a function in R^1 we denote by $\text{var } [f; (a, b)]$ its variation on $(a, b) = R^1 \cap \{t : a < t < b\}$. If f is known to be summable over every compact subset in (a, b) , we shall use $\text{var ess } [f; (a, b)]$ to denote $\sup_{\psi} \int_a^b \psi'(t)f(t) dt$, ψ ranging over all infinitely differentiable functions with compact $\text{spt } \psi \subset (a, b)$ such that $|\psi| \leq 1$.

REMARK. It follows easily from the Riesz representation theorem and elementary distribution theory that $\text{var ess } [f; (a, b)] < \infty$ implies the existence of a function g in (a, b) such that $g = f$ a.e. in (a, b) and $\text{var } [g; (a, b)] = \text{var ess } [f; (a, b)]$.

Clearly, $\text{var } [f; (a, b)] = \text{var ess } [f; (a, b)]$ whenever f is continuous in (a, b) .

1.9. LEMMA. If c_M is the characteristic function of a Borel set $M \subset R^1$, then $\text{var ess } [c_M; (a, b)]$ equals the number of hits of (a, b) on M .

Proof. Let q stand for the number of all hits of (a, b) on M . If $q < \infty$ and $a_1 < \dots < a_q$ are all the hits, then no (a_j, a_{j+1}) can meet both M and $(a, b) - M$ in a set of positive linear measure. It follows that either M or $(a, b) - M$ is equivalent with $\bigcup_k (a_{2k-1}, a_{2k})$, where $1 \leq k, 2k \leq q$. Consequently, $\text{var ess } [c_M; (a, b)] = q$. Conversely, if $\text{var ess } [c_M; (a, b)] < \infty$, then there is a g with $\text{var } [g; (a, b)] < \infty$ such that $g = c_M$ a.e. in (a, b) .

Clearly, this implies $q < \infty$.

1.10. LEMMA. Let f be a bounded Baire function in R^m , $y \in R^m$, $0 \leq a < b \leq \infty$. For $\theta \in \Gamma$ put

$$(1.4) \quad f_\theta(t) = f(y + t\theta), \quad t \in R^1.$$

Then $\text{var ess } [f_\theta; (a, b)]$ is a Baire function of the variable θ on Γ and the integral

$$\int_\Gamma \text{var ess } [f_\theta; (a, b)] dH_{m-1}(\theta)$$

equals

$$v(a, b, f) = \sup_\psi \int_{R^m} f(x) \text{grad } \psi(x) \cdot \frac{y-x}{|y-x|^m} dx,$$

ψ ranging over all functions in D with

$$(1.5) \quad \text{spt } \psi \subset R^m \cap \{x : a < |x-y| < b\}, \quad |\psi| \leq 1.$$

Proof. We may assume $y=0$, $b < \infty$. Using the notation from (1.4) we obtain for any $\psi \in D$ satisfying (1.5)

$$\begin{aligned} \int_{R^m} f(x) \text{grad } \psi(x) \cdot \frac{x}{|x|^m} dx &= \int_\Gamma \left(\int_a^b f_\theta(t) \psi'_\theta(t) dt \right) dH_{m-1}(\theta), \\ \int_a^b f_\theta(t) \psi'_\theta(t) dt &\leq \text{var ess } [f_\theta; (a, b)]. \end{aligned}$$

Assuming that we know already that $\text{var ess } [f_\theta; (a, b)]$ is measurable (H_{m-1}) on Γ we get

$$v(a, b, f) \leq \int_\Gamma \text{var ess } [f_\theta; (a, b)] dH_{m-1}(\theta).$$

It remains to prove that $\text{var ess } [f_\theta; (a, b)]$ is a Baire function of θ and

$$(1.6) \quad \int_\Gamma \text{var ess } [f_\theta; (a, b)] dH_{m-1}(\theta) \leq v(a, b, f).$$

To show this we first assume, in addition, that

(I). f_θ has a continuous derivative on (a, b) for every $\theta \in \Gamma$ and

$$\sup \{|f'_\theta(t)| : \theta \in \Gamma, c < t < d\} = K(c, d) < \infty$$

whenever $a < c < d < b$.

For every positive integer N we subdivide (a, b) by means of points

$$a_k = a_k^N = a + k2^{-N}(b-a), \quad 1 \leq k < 2^N.$$

Consider $k < 2^N - 2$. Since $\text{sign} [f_\theta(a_{k+1}) - f_\theta(a_k)]$ is a Baire function of θ , there are functions $\phi_{ks} \in D$ such that $|\phi_{ks}| \leq 1$ and

$$\lim_{s \rightarrow \infty} \phi_{ks}(\theta) = \text{sign} [f_\theta(a_{k+1}) - f_\theta(a_k)] \quad \text{a.e. } (H_{m-1})$$

on Γ . Further express the characteristic function of (a_k, a_{k+1}) as $\lim_{s \rightarrow \infty} \rho_{ks}$, where ρ_{ks} are infinitely differentiable functions in R^1 with

$$\text{spt } \rho_{ks} \subset (a_k, a_{k+1}), \quad |\rho_{ks}| \leq 1,$$

and define

$$\psi_s(t\theta) = - \sum_{k=1}^{2^N-2} \phi_{ks}(\theta) \rho_{ks}(t), \quad t \geq 0, \theta \in \Gamma.$$

Then

$$\psi_s \in D, \quad |\psi_s| \leq 1, \quad \text{spt } \psi_s \subset R^m \cap \{x : a < |x| < b\}.$$

Consequently,

$$v(a, b, f) \geq \int_{\Gamma} \left[\int_a^b f_\theta(t) \psi'_{s\theta}(t) dt \right] dH_{m-1}(\theta).$$

The sequence of integrals

$$\int_a^b f_\theta(t) \psi'_{s\theta}(t) dt = \sum_{k=1}^{2^N-2} \phi_{ks}(\theta) \int_{a_k}^{a_{k+1}} \rho_{ks}(t) f'_\theta(t) dt$$

is dominated by $(b-a)K(a_1, a_{2^N-1})$ and converges, as $s \rightarrow \infty$, to

$$\sigma_N(\theta) = \sum_{k=1}^{2^N-2} |f_\theta(a_{k+1}) - f_\theta(a_k)|$$

a.e. (H_{m-1}) on Γ . Hence we conclude

$$v(a, b, f) \geq \int_{\Gamma} \sigma_N(\theta) dH_{m-1}(\theta).$$

Noting that $\sigma_N(\theta) \uparrow \text{var} [f_\theta; (a, b)]$ as $N \rightarrow \infty$ we see that $\text{var} [f_\theta; (a, b)]$ is a Baire function of θ and (1.6) holds in this special case.

Let us now drop the additional assumptions (I) on f . For every positive integer N we fix a symmetric infinitely differentiable function ω_N in R^1 with

$$\text{spt } \omega_N \subset (-1/N, 1/N), \quad \int_{R^1} \omega_N(t) dt = 1$$

and define f_N so that $f_{N\theta} = f_\theta * \omega_N$ (=the convolution of f_θ and ω_N) on the positive real axis, $f_N(0) = 0$. Let $a^N = a + 1/N$, $b^N = b - 1/N$, $2/N < b - a$. It follows from the first part of the proof that

$$(1.7) \quad \int_{\Gamma} \text{var } [f_{N\theta}; (a^N, b^N)] dH_{m-1}(\theta) = v(a^N, b^N, f_N).$$

If ψ_N is obtained from ψ in the same way as f_N from f , then

$$\psi \in D, |\psi| \leq 1, \text{ spt } \psi \subset R^m \cap \{x : a^N < |x| < b^N\}$$

imply

$$\psi_N \in D, |\psi_N| \leq 1, \text{ spt } \psi_N \subset R^m \cap \{x : a < |x| < b\}$$

and

$$\int_{a^N}^{b^N} \psi'_\theta(t) f_{N\theta}(t) dt = \int_a^b \psi'_{N\theta}(t) f_\theta(t) dt.$$

Consequently,

$$(1.8) \quad v(a^N, b^N, f_N) \leq v(a, b, f).$$

The same argument shows that

$$(1.9) \quad \text{var ess } [f_{N\theta}; (a^N, b^N)] \leq \text{var ess } [f_\theta; (a, b)].$$

It is easy to see that

$$\liminf_{N \rightarrow \infty} \text{var ess } [f_{N\theta}; (a^N, b^N)] \geq \text{var ess } [f_\theta; (a, b)],$$

which together with (1.9) yields

$$(1.10) \quad \lim_{N \rightarrow \infty} \text{var ess } [f_{N\theta}; (a^N, b^N)] = \text{var ess } [f_\theta; (a, b)].$$

In particular, $\text{var ess } [f_\theta; (a, b)]$ is a Baire function of θ . (1.7), (1.8), and (1.10) imply (1.6).

REMARK. The above lemma could also be derived from general theorems on functions, whose partial derivatives are measures; cf. [FL], [KR], [P] on the subject.

Now it is easy to present the following.

1.11. **Proof of Proposition 1.6.** Let f be the characteristic function of G . By 1.9 and 1.10

$$\text{var ess } [f_\theta; (0, r)] = n_r(\theta, y),$$

$$v_r(y) = v(0, r, f).$$

If $n_r(\theta, y) < \infty$, then $G \cap \{y + t\theta : 0 < t < r\}$ is equivalent (H_1) with a finite union

of disjoint segments, whose end points are hits of $\{y+t\theta : 0 < t < r\}$ on G and, possibly, y and $y+r\theta$. Hence we conclude for $\psi \in E_r(y)$

$$\left| \int_0^\infty f_\theta(t) \psi'_\theta(t) dt \right| \leq 1 + n_r(\theta, y), \quad \theta \in \Gamma,$$

$$\int_G \text{grad } \psi(x) \cdot (y-x) |y-x|^{-m} dx \leq \int_\Gamma [1 + n_r(\theta, y)] dH_{m-1}(\theta) = A + v_r(y).$$

It remains to note that, in case $y \in B$ and G is open, $NU\delta_y$ is a measure if and only if

$$\sup \{ \langle \psi, NU\delta_y \rangle : \psi \in D, |\psi| \leq 1 \} < \infty.$$

1.12. REMARK. Let us observe that, in case $y \in B$ and $NU\delta_y \in C^*(B)$,

$$(1.11) \quad v_\infty(y) \leq \|NU\delta_y\| \leq A + v_\infty(y).$$

Now we are in position to answer the question raised in 1.2.

1.13. THEOREM. $NU\mu$ is a measure for every $\mu \in C^*(B)$ if and only if

$$(1.12) \quad V = \sup_{y \in B} v_\infty(y) < \infty.$$

If this is the case, then

$$NU : \mu \rightarrow NU\mu$$

is a bounded linear operator on $C^*(B)$,

$$\|NU\| \leq A + V$$

and (1.2) holds for every bounded Baire function ψ on B . In particular,

$$NU\mu(M) = \int_B NU\delta_y(M) d\mu(y)$$

for $\mu \in C^*(B)$ and every Borel set $M \subset B$.

Proof. With every $\psi \in D$ we associate a linear functional L_ψ over $C^*(B)$ defined by

$$\langle \mu, L_\psi \rangle = \langle \psi, NU\mu \rangle, \quad \mu \in C^*(B).$$

Denoting

$$P_\psi = G \cap \text{spt } \psi, \quad s_\psi = \sup |\text{grad } \psi|,$$

we obtain from (1.1)

$$|\langle \mu, L_\psi \rangle| \leq s_\psi A \text{diam}(B \cup P_\psi) \|\mu\|$$

which shows that every L_ψ is bounded on $C^*(B)$. Let $E = D \cap \{\psi : |\psi| \leq 1\}$. Then $NU\mu$ is a measure if and only if

$$\sup_{\psi \in E} \langle \psi, NU\mu \rangle < \infty.$$

In particular, if $NU\mu$ is a measure for every $\mu \in C^*(B)$, then the class of functionals $\{L_\psi\}_{\psi \in E}$ must be pointwise bounded on $C^*(B)$ and, by the uniform boundedness principle,

$$\sup_{\psi \in E} \|L_\psi\| = K < \infty.$$

Employing (1.11) we get for every $y \in B$

$$v_\infty(y) \leq \sup_{\psi \in E} \langle \psi, NU\delta_y \rangle \leq K.$$

Conversely, if (1.12) holds, then (1.2) together with (1.11) imply

$$\sup_{\psi \in E} |\langle \psi, NU\mu \rangle| \leq (A + V)\|\mu\|$$

for every $\mu \in C^*(B)$. It is also easily seen that in this case (1.2) extends to any bounded Baire function ψ .

2. Double layer potentials.

2.1. *Notation.* Throughout this paragraph $C \subset R^m$ will denote a Borel set with a compact boundary B . Given $z \in R^m$ we put

$$D(z) = D \cap \{\psi : z \notin \text{spt } \psi\}$$

and define

$$(2.1) \quad W_\psi(z) = \int_C \text{grad } \psi(x) \cdot \frac{x-z}{|x-z|^m} dx, \quad \psi \in D(z).$$

If it is necessary to specify C we write W_ψ^C instead of W_ψ . In case C has a smooth boundary with exterior normal n the integral (2.1) reduces to

$$\int_B \psi(y) \frac{(y-z) \cdot n(y)}{|y-z|^m} dH_{m-1}(y),$$

which is the classical double-layer potential. If ψ vanishes in some neighborhood of B then there is a $Q \subset R^m$ with a smooth boundary such that

$$\text{spt } \psi \cap C \subset \text{int } Q, \quad \text{cl } Q \subset \text{int } C,$$

whence

$$W_\psi(z) = W_\psi^Q(z) = 0.$$

If $z \notin B$, we use this observation to extend $W_\psi(z)$ from $D(z)$ to D defining

$$W_\psi(z) = W_{\tilde{\psi}}(z),$$

where $\tilde{\psi}$ is an arbitrary function in $D(z)$ coinciding with given $\psi \in D$ in some neighborhood of B . $W_\psi(z)$ may thus be considered as a distribution over D with support in B (compare [D, Chapter III, p. 157]).

For fixed $\psi \in D$, $W_\psi(z)$ is a harmonic function of z in $R^m - B$. Indeed, if O is an open set with $B \cap \text{cl } O = \emptyset$, then there is a $\tilde{\psi} \in D$ coinciding with ψ in some neighborhood of B and vanishing on O ; clearly,

$$W_\psi(z) = W_{\tilde{\psi}}(z) = \int_{C-O} \text{grad } \tilde{\psi}(x) \cdot \frac{x-z}{|x-z|^m} dx$$

is a harmonic function of z in O .

Our main objective in this paragraph is to find necessary and sufficient geometric conditions on C securing natural extendability of W_ψ from D to broader class of continuous functions and also "nice behaviour" (e.g., boundedness) of W_ψ near B for each continuous ψ .

2.2. LEMMA. Fix $z \in R^m$. Then

$$(2.2) \quad v_\infty^C(z) < \infty$$

is a necessary and sufficient condition to secure

$$\lim_{k \rightarrow \infty} W_{\psi_k}(z) = W_\psi(z)$$

for every sequence of $\psi_k \in D(z)$ converging uniformly (as $k \rightarrow \infty$) to $\psi \in D(z)$. If (2.2) holds then there is a $\nu_z \in C^*(B)$ such that

$$(2.3) \quad W_\psi(z) = \int_B \psi(y) d\nu_z(y), \quad \psi \in D(z),$$

$$(2.4) \quad \nu_z(\{z\}) = 0,$$

$$(2.5) \quad \|\nu_z\| = v_\infty^C(z).$$

(2.3) together with any of the two conditions (2.4), (2.5) determine ν_z uniquely.

Proof. This follows at once from the equality

$$(2.6) \quad v_\infty^C(z) = \sup \{W_\psi(z) : \psi \in D(z), |\psi| \leq 1\}$$

established in 1.6.

2.3. REMARK. If (2.2) holds we extend $W \cdots (z)$ defining

$$Wf(z) = \int_B f(y) d\nu_z(y)$$

for any bounded Baire function f on B .

In order to present another integral representation for $Wf(z)$ we introduce the following.

2.4. Notation. Fix $z \in R^m$ and $\theta \in \Gamma$. We put for $t > 0$

$$s(t; z, \theta) = \sigma (= \pm 1)$$

if there is a $\delta > 0$ such that

$$z + (t + \sigma\tau)\theta \in R^m - C, \quad z + (t - \sigma\tau)\theta \in C$$

for a.e. $\tau \in (0, \delta)$; otherwise we set $s(t; z, \theta) = 0$.

Clearly, $s(t; z, \theta) \neq 0$ only if $z + t\theta$ is a hit of $\{z + \tau\theta : \tau > 0\}$ on C .

2.5. LEMMA. If $v_\infty^C(z) < \infty$ then

$$(2.7) \quad Wf(z) = \int_\Gamma \left\{ \sum_{t>0} f(z+t\theta) s(t; z, \theta) \right\} dH_{m-1}(\theta)$$

for any bounded Baire function f on B .

Proof. Let $v_\infty^C(z) < \infty$. If $f \in D(z)$ then

$$\begin{aligned} Wf(z) &= \int_C \text{grad } f(x) \cdot \frac{x-z}{|x-z|^m} dx \\ &= \int_\Gamma \left\{ \int_{C_\theta} \partial_\theta f(z+t\theta) dt \right\} dH_{m-1}(\theta), \end{aligned}$$

where

$$(2.8) \quad C_\theta = \{t : t > 0, z+t\theta \in C\}, \quad \partial_\theta f = \theta \cdot \text{grad } f.$$

Noting that $n_\infty^C(\theta, z) < \infty$ implies

$$\int_{C_\theta} \partial_\theta f(z+t\theta) dt = \sum_{t>0} f(z+t\theta) s(t; z, \theta)$$

we obtain (2.7).

If $\{f_k\}$ is a pointwise convergent sequence of functions on B such that, for all k , $|f_k| \leq K$ and (2.7) holds with f replaced by f_k , then

$$\left| \sum_{t>0} f_k(z+t\theta) s(t; z, \theta) \right| \leq K n_\infty^C(\theta, z)$$

a.e. $\{H_{m-1}\}$ on Γ and, by the Lebesgue convergence theorem, (2.7) holds for $f = \lim_k f_k$ as well.

We conclude that (2.7) is valid for every bounded Baire function f vanishing at z ; in view of (2.4), vanishing at z is irrelevant.

2.6. PROPOSITION. Let $v_\infty^C(z) < \infty$. Denote by K_z and L_z the set of all $\theta \in \Gamma$ for which there is an $\varepsilon = \varepsilon(\theta) > 0$ such that

$$H_1(\{z+t\theta : 0 < t < \varepsilon\} \cap C) = 0$$

and

$$H_1(\{z+t\theta : 0 < t < \varepsilon\} - C) = 0,$$

respectively. Then K_z, L_z are measurable (H_{m-1}),

$$(2.9) \quad H_{m-1}(\Gamma - (K_z \cup L_z)) = 0$$

and $\nu_z(B) = H_{m-1}(L_z)$ or $\nu_z(B) = -H_{m-1}(K_z)$ according as C is bounded or not. If $\psi \in D$, then

$$(2.10) \quad \int_C \operatorname{grad} \psi(x) \cdot \frac{x-z}{|x-z|^m} dx = W_\psi(z) - H_{m-1}(L_z)\psi(z).$$

If Q is a convex Borel set, then

$$(2.11) \quad |\nu_z(B \cap Q)| \leq A.$$

Proof. It is easily seen that

$$\Gamma \cap \{\theta : n_\infty^C(\theta, z) < \infty\} \subset L_z \cup K_z$$

whence (2.9) follows at once.

Fix now a $\theta \in \Gamma$ with $n_\infty^C(\theta, z) < \infty$. Let

$$(2.12) \quad t_1 < \dots < t_q$$

be all the points $t \in (0, \infty)$ with $s(t; z, \theta) \neq 0$ (cf. 2.4). Clearly,

$$(2.13) \quad s(t_{j+1}; \dots) = -s(t_j; \dots), \quad 1 \leq j < q$$

and $s(t_1; \dots) = 1$ or $s(t_1; \dots) = -1$ according as $\theta \in L_z$ or $\theta \in K_z$. If C is bounded, then $s(t_q; \dots) = 1$, while $s(t_q; \dots) = -1$ in the opposite case. We conclude that

$$\sum (\theta) = \sum_{t>0} s(t; z, \theta)$$

almost (H_{m-1}) equals the characteristic function of L_z if C is bounded, while $-\sum (\theta)$ almost equals the characteristic function of K_z in the opposite case. Employing (2.7) with $f \equiv 1$ we get the first part of our proposition.

Let now f be the characteristic function of a convex Borel set Q . Consider again a fixed $\theta \in \Gamma$, $n_\infty^C(\theta, z) < \infty$, and the corresponding sequence (2.12). If t_i and t_k are the first and the last members of (2.12) with $z + t_j\theta \in Q$, respectively, then (2.13) implies

$$\left| \sum_{j=1}^q f(z + t_j\theta) s(t_j; z, \theta) \right| = \left| \sum_{j=i}^k s(t_j; z, \theta) \right| \leq 1,$$

whence (2.11) follows by 2.5. If $\psi \in D$ then we have with the notation from (2.8)

$$\int_{C_\theta} \partial_\theta \psi(z + t\theta) dt = \sum_{t>0} \psi(z + t\theta) s(t; z, \theta)$$

for $\theta \in K_z \cap \{\theta : n_\infty^C(\theta, z) < \infty\}$, while

$$\int_{C_\theta} \partial_\theta \psi(z + t\theta) dt = \sum_{t>0} \psi(z + t\theta) s(t; z, \theta) - \psi(z)$$

for $\theta \in L_z \cap \{\theta : n_\infty^c(\theta, z) < \infty\}$. Hence

$$\begin{aligned} \int_C \operatorname{grad} \psi(x) \cdot \frac{x-z}{|x-z|^m} dx &= \int_\Gamma \left(\int_{C_\theta} \partial_\theta \psi(z+t\theta) dt \right) dH_{m-1}(\theta) \\ &= W\psi(z) - H_{m-1}(L_z)\psi(z). \end{aligned}$$

2.7. LEMMA. Let $v_\infty^c(z) < \infty$ and define L_z as in 2.6. If $M \subset \Gamma$ is measurable (H_{m-1}), $H_{m-1}(M) > 0$ and

$$\Lambda_M = \{z + t\theta : \theta \in M, t > 0\},$$

then

$$(2.14) \quad \lim_{r \rightarrow 0+} \frac{H_m(\Omega_r(z) \cap C \cap \Lambda_M)}{H_m(\Omega_r(z) \cap \Lambda_M)} = \frac{H_{m-1}(L_z \cap M)}{H_{m-1}(M)}.$$

In particular, C has an m -dimensional density

$$d_C(z) = H_{m-1}(L_z)/A$$

at z .

Proof. Let $\varepsilon(\theta)$ have the meaning described in the definition of K_z, L_z in 2.6 and put

$$K^r = M \cap \{\theta : \theta \in K_z, \varepsilon(\theta) > r\},$$

$$L^r = M \cap \{\theta : \theta \in L_z, \varepsilon(\theta) > r\}.$$

We have

$$H_m(\Omega_r(z) \cap C \cap \Lambda_M) \geq m^{-1}r^m \operatorname{inn} H_{m-1}(L^r),$$

$$H_m((\Omega_r(z) - C) \cap \Lambda_M) \geq m^{-1}r^m \operatorname{inn} H_{m-1}(K^r),$$

where $\operatorname{inn} H_{m-1}$ stands for the inner $(m-1)$ -dimensional Hausdorff measure. Denoting

$$d_r = \frac{H_m(\Omega_r(z) \cap C \cap \Lambda_M)}{H_m(\Omega_r(z) \cap \Lambda_M)}$$

and noting that

$$K^r \uparrow (K_z \cap M), \quad L^r \uparrow (L_z \cap M)$$

as $r \downarrow 0$, we obtain

$$\liminf_{r \rightarrow 0+} d_r \geq H_{m-1}(L_z \cap M)/H_{m-1}(M),$$

$$\liminf_{r \rightarrow 0+} (1 - d_r) \geq H_{m-1}(K_z \cap M)/H_{m-1}(M),$$

whence (2.14) follows by (2.9).

2.8. *Notation.* $P(C)$ will denote the perimeter of C defined by

$$P(C) = \sup_w \int_C \operatorname{div} w(x) \, dx,$$

where $w = [w_1, \dots, w_m]$ ranges over all vector-valued functions with m components $w_j \in D$ satisfying

$$\left(\sum_{j=1}^m w_j^2 \right)^{1/2} = |w| \leq 1.$$

(Further information on sets with finite perimeter may be found in [DG], [F3], [FL], [MA].)

For $M \subset R^m$ and $z \in R^m$ we let

$$\operatorname{dist}(z, M) = \inf \{|z - y| : y \in M\}.$$

2.9. **LEMMA.** $v_\infty^C(z)$ is a lower semicontinuous function of z on R^m satisfying the inequality

$$v_\infty^C(z) \leq P(C)(\operatorname{dist}(z, B))^{1-m}, \quad z \notin B.$$

Proof. If $K < v_\infty^C(z)$, then there is a $\psi \in D(z)$ such that $|\psi| \leq 1$ and $W_\psi(z) > K$ (see (2.6)). Hence

$$\liminf_{y \rightarrow z} v_\infty^C(y) \geq \lim_{y \rightarrow z} W_\psi(y) = W_\psi(z) > K.$$

Suppose now that $z \notin B$, fix an arbitrary $\psi \in D(z)$ with $|\psi| \leq 1$ and a positive $\rho < \operatorname{dist}(z, B)$. Then there is a $\tilde{\psi} \in D$, $|\tilde{\psi}| \leq 1$, which coincides with ψ in some neighborhood of B and vanishes on $\Omega_\rho(z)$. Let us define $w(z) = O \in R^m$,

$$w(x) = \tilde{\psi}(x) \frac{x - z}{|x - z|^m}, \quad x \neq z,$$

and observe that $|w| \leq \rho^{1-m}$,

$$\operatorname{grad} \tilde{\psi}(x) \cdot \frac{x - z}{|x - z|^m} = \operatorname{div} w(x).$$

Consequently,

$$W_\psi(z) = W_{\tilde{\psi}}(z) = \int_C \operatorname{div} w(x) \, dx \leq \rho^{1-m} P(C).$$

REMARK. We see that $v_\infty^C(z)$ is finite on $R^m - B$ provided $P(C) < \infty$. The converse is also true as it follows from the following

2.10. **PROPOSITION.** *If*

$$\sum_{j=1}^{m+1} v_\infty^C(z_j) < \infty$$

for an $(m+1)$ -tuple of points $z_1, \dots, z_{m+1}$ in general position (i.e., not situated on a single hyperplane), then

$$(2.15) \quad P(C) < \infty.$$

Proof. To prove (2.15) it is sufficient to show that

$$\sup \left\{ \int_C \partial_\theta \psi(x) dx : \psi \in D, |\psi| \leq 1 \right\} < \infty$$

for every $\theta \in \Gamma$. Fix $\theta \in \Gamma$. Let Π_j denote the hyperplane determined by $\{z_k : k \neq j\}$. Since

$$\bigcup_{j=1}^{m+1} (R^m - \Pi_j) = R^m,$$

there are $\alpha_j \in D$ such that

$$\Pi_j \cap \text{spt } \alpha_j = \emptyset$$

and

$$\alpha = \sum_{j=1}^{m+1} \alpha_j = 1$$

in some neighborhood of B .

Noting that

$$\int_C \alpha(x) \partial_\theta \psi(x) dx = \int_C \partial_\theta \psi(x) dx$$

we see that it is sufficient to prove that

$$\sup \left\{ \int_C \alpha_j(x) \partial_\theta \psi(x) dx : \psi \in D, |\psi| \leq 1 \right\} < \infty$$

for $j=1, \dots, m+1$. Consider, for instance, $j=1$. If $x \in \text{spt } \alpha_1$, then $x - z_2, \dots, x - z_{m+1}$ are linearly independent. Consequently,

$$\theta = \sum_{k=2}^{m+1} a_k(x) \frac{x - z_k}{|x - z_k|^m},$$

where a_k are infinitely differentiable in some neighborhood of $\text{spt } \alpha_1$. Extending a_k arbitrarily to R^m we get

$$\int_C \alpha_1(x) \partial_\theta \psi(x) dx = \sum_{k=2}^{m+1} \int_C \alpha_1(x) a_k(x) \text{grad } \psi(x) \cdot \frac{x - z_k}{|x - z_k|^m} dx.$$

Fix $k \in \langle 2, m+1 \rangle$ and define $F(x) = \alpha_1(x) a_k(x)$. Then $F \in D(z_k)$ and denoting $K = \max |F|$ we obtain for any $\psi \in D$ with $|\psi| \leq 1$

$$\int_C F(x) \text{grad } \psi(x) \cdot \frac{x - z_k}{|x - z_k|^m} dx = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \int_C \operatorname{grad} (F(x)\psi(x)) \cdot \frac{x-z_k}{|x-z_k|^m} dx \leq K v_\infty^C(z_k), \\ I_2 &= - \int_C \psi(x) \operatorname{grad} F(x) \cdot \frac{x-z_k}{|x-z_k|^m} dx \\ &\leq \int_C |\operatorname{grad} F(x)| \cdot |x-z_k|^{1-m} dx < \infty. \quad (2) \end{aligned}$$

2.11. REMARK. It follows from 2.2, 2.9, and 2.10 that (2.12) is a necessary and sufficient condition to secure continuous dependence (with respect to uniform convergence) of $W\psi(z)$ on ψ for every $z \notin B$. For this reason we agree to impose (2.15) on C throughout the rest of the present paragraph.

Let us recall that $\theta \in \Gamma$ is called the exterior normal of C at y in the sense of Federer provided the symmetric difference of C and the half-space

$$R^m \cap \{x : (x-y) \cdot \theta < 0\}$$

has m -dimensional density 0 at y (cf. [F1]).

In what follows the term exterior normal is always to be interpreted in this sense. We put $n^C(y) = n(y) = \theta$ if θ is the exterior normal of C at y ; otherwise $n(y)$ denotes the zero vector. The set of all y with $n(y) \neq 0$ is called the reduced boundary of C and will be denoted by $\hat{B}$. It is known from [DG2] and [F3] that

$$H_{m-1}(\hat{B}) < \infty$$

and

$$\int_C \operatorname{div} w(x) dx = \int_B w(y) \cdot n(y) dH_{m-1}(y)$$

for every vector-valued function $w = [w_1, \dots, w_m]$ with components $w_j \in D$.

2.12. LEMMA. For every $z \in R^m$

$$(2.16) \quad v_\infty^C(z) = \int_B \frac{|n(y) \cdot (y-z)|}{|y-z|^m} dH_{m-1}(y).$$

If $v_\infty^C(z) < \infty$ and $M \subset B$ is a Borel set, then

$$v_z(M) = \int_M \frac{n(y) \cdot (y-z)}{|y-z|^m} dH_{m-1}(y).$$

Proof. Fix $z \in R^m$. Let $\psi \in D(z)$ and put $w(z) = 0$ ($\in R^m$),

$$w(x) = \psi(x) \frac{x-z}{|x-z|^m}, \quad x \neq z.$$

(2) The author is indebted to Herbert Federer for simplification of this proof.

Then

$$W_w(z) = \int_C \operatorname{div} w(x) dx = \int_B \psi(y) \frac{n(y) \cdot (y-z)}{|y-z|^m} dH_{m-1}(y)$$

and (2.16) follows from (2.6). Let now $v_\infty^C(z) < \infty$. As we have just seen,

$$\int_B f dv_z = \int_B f(y) \frac{n(y) \cdot (y-z)}{|y-z|^m} dH_{m-1}(y)$$

provided $f \in D(z)$; it is easily seen that this formula extends to any bounded Baire function f .

The following result will be useful below:

2.13. THEOREM. *Let*

$$V^C = \sup \{v_\infty^C(y) : y \in B\}.$$

Then $v_\infty^C(z) \leq A + V^C$ for every $z \in R^m$.

Proof. We may assume $V^C < \infty$. Fix $z \in R^m - B$ and let d be an arbitrary number less than $v_\infty^C(z)$. Then there exist mutually disjoint closed parallelepipeds $K_1, \dots, K_q$ such that

$$\sum_{j=1}^q |\nu_z(B \cap K_j)| > d.$$

Put $\sigma_j = \operatorname{sign} \nu_z(B \cap K_j)$ and consider the function

$$h(x) = \sum_{j=1}^q \sigma_j \nu_x(B \cap K_j),$$

which is harmonic on

$$R^m - \bigcup_{j=1}^q B \cap K_j \supset R^m - B.$$

Fix an arbitrary $y \in B$. If $y \notin \bigcup_{j=1}^q K_j$, then

$$\lim_{x \rightarrow y} h(x) = h(y) \leq \|\nu_y\| \leq V^C.$$

In the opposite case we may assume that $y \in K_1$, so that

$$\lim_{x \rightarrow y} \sum_{j=2}^q \sigma_j \nu_x(B \cap K_j) = \sum_{j=2}^q \sigma_j \nu_y(B \cap K_j) \leq \|\nu_y\| \leq V^C$$

and, by Proposition 2.6,

$$\sup_x |\nu_x(B \cap K_1)| \leq A.$$

We see that

$$\limsup_{x \rightarrow y: x \notin B} h(x) \leq A + V^C.$$

Noting that $h(x) \rightarrow 0$ as $|x| \rightarrow \infty$ we conclude that $h \leq A + V^C$ on $R^m - B$. In particular, $d < h(z) \leq A + V^C$.

2.14. COROLLARY. *If $r > 0$ and $z \in R^m$, then*

$$(2.17) \quad H_{m-1}(\Omega_r(z) \cap \hat{B}) \leq m(m+1)^m(A + V^C)r^{m-1}.$$

Proof. To prove (2.17) we may clearly assume that $z=0$. Noting that V^C is invariant with respect to dilations of C we observe that it is sufficient to establish (2.17) for $r=1$ only. Let e^i denote the point in R^m all of whose coordinates vanish with the exception of the i th one which is equal to $m+1$. We have then for $\theta \in \Gamma$ and $y \in \Omega = \Omega_1(0)$

$$\sum_{i=1}^m |\theta \cdot (y - e^i)| \geq 1,$$

so that

$$\begin{aligned} H_{m-1}(\hat{B} \cap \Omega) &\leq \sum_{i=1}^m \int_B |n(y) \cdot (y - e^i)| dH_{m-1}(y) \\ &\leq (m+1)^m \sum_{i=1}^m \int_B \frac{|n(y) \cdot (y - e^i)|}{|y - e^i|^m} dH_{m-1}(y) \\ &= (m+1)^m \sum_{i=1}^m v_\omega^C(e^i) \leq m(m+1)^m(A + V^C). \end{aligned}$$

2.15. THEOREM. *Let $C(B)$ denote the Banach space of all continuous functions f on B with the norm $\|f\| = \sup |f|$. If Wf is bounded on $R^m - B$ for every $f \in C(B)$ then*

$$(2.18) \quad V^C < \infty.$$

If

$$C_i = R^m \cap \{z : d_C(z) = i\} \quad (i = 0, 1)$$

and (2.18) holds, then Wf is bounded and uniformly continuous on each of the sets C_0, C_1 and

$$(2.19) \quad \lim_{z \rightarrow y; z \in C_1} Wf(z) = Wf(y) + A(1 - d_C(y))f(y) \quad \text{for } y \in B \cap \text{cl } C_1,$$

$$(2.20) \quad \lim_{z \rightarrow y; z \in C_0} Wf(z) = Wf(y) - Ad_C(y)f(y) \quad \text{for } y \in B \cap \text{cl } C_0$$

whenever $f \in C(B)$.

Proof. If $Wf(z) = \langle f, v_z \rangle$ is a bounded function of z on $Q \subset R^m$ for every $f \in C(B)$ then, by the uniform boundedness principle, $\|v_z\| = v_\omega^C(z)$ is bounded on Q . In view of 2.9, $v_\omega^C(z)$ must be bounded on $\text{cl } Q$ as well. For $Q = R^m - B$ we get the first part of our theorem. Assume (2.18) and fix $y \in B$. If $f \equiv 1$ on B then (2.19),

(2.20) follow from 2.6, 2.7. It is therefore sufficient to prove (2.19), (2.20) assuming $f \in C(B)$, $f(y)=0$. For every k we have the decomposition $f=f_k+g_k$, where $f_k \in C(B)$ vanishes in some neighborhood of y in B and $\|g_k\| \leq 1/k$. Then Wf_k is continuous at y and $|Wg_k| \leq (A+V^c)/k$. We see that $Wf=\lim_{k \rightarrow \infty} Wf_k$ is continuous at y . The rest is obvious.

3. The Fredholm radius of an operator.

3.1. *Notation.* As in the introduction, G will stand for a fixed open set with a compact boundary B in R^m . We put $C=R^m-G$ and write $v_r(y)=v_r^G(y)$ ($=v_r^C(y)$), $V=V^c$ (cf. 1.6, 1.7, (1.12), 2.13). We always assume

$$(3.1) \quad V < \infty.$$

In view of 1.13,

$$(3.2) \quad NU : \mu \rightarrow NU\mu$$

is a bounded linear operator on $C^*(B)$. By 2.7, G has an m -dimensional density $d_G(y)$ at any $y \in R^m$.

3.2. **LEMMA.** *If f is a bounded Baire function on B then*

$$(3.3) \quad \langle f, NU\delta_y \rangle = Ad_G(y)f(y) + W^c f(y), \quad y \in B.$$

Proof. It is sufficient to prove (3.3) for $f \in D$ only. Employing 1.3, 2.6, and 2.7 we obtain

$$\begin{aligned} \langle f, NU\delta_y \rangle &= \int_G \text{grad } f(x) \cdot \frac{y-x}{|y-x|^m} dx \\ &= Af(y) + \int_C \text{grad } f(x) \cdot \frac{x-y}{|x-y|^m} dx \\ &= Ad_G(y)f(y) + W^c f(y). \end{aligned}$$

3.3. **DEFINITION.** If $f \in C(B)$ we define

$$(3.4) \quad \overline{W}f(y) = \langle f, NU\delta_y \rangle - \frac{1}{2}Af(y), \quad y \in B.$$

3.4. **LEMMA.** $\overline{W}f \in C(B)$ whenever $f \in C(B)$. The operator

$$(3.5) \quad \overline{W} : f \rightarrow \overline{W}f$$

is bounded on $C(B)$ and the operator (3.2) is adjoint to $\frac{1}{2}AI + \overline{W}$, where I is the identity operator on $C(B)$. If $f \in C(B)$ and C_1 has the meaning described in 2.15, then

$$(3.6) \quad \overline{W}f(y) = \lim_{z \rightarrow y; z \in C_1} W^c f(z) - \frac{1}{2}Af(y), \quad y \in B \cap \text{cl } C_1,$$

$$\begin{aligned} (3.7) \quad \overline{W}f(y) &= W^c f(y) + A(d_G(y) - \frac{1}{2})f(y) \\ &= \lim_{z \rightarrow y; z \in G} W^c f(z) + \frac{1}{2}Af(y), \quad y \in B. \end{aligned}$$

Proof. (3.7), (3.6) follow from (3.4), (3.3), and (2.19), (2.20). By (3.7), $\overline{W}f \in C(B)$ for $f \in C(B)$. If v_y has the meaning described in 2.2 and

$$(3.8) \quad \bar{v}_y = A(d_G(y) - \tfrac{1}{2})\delta_y + v_y,$$

then

$$(3.9) \quad \overline{W}f(y) = \langle f, \bar{v}_y \rangle, \quad f \in C(B), \quad y \in B,$$

whence

$$(3.10) \quad \|\overline{W}\| = \sup_{y \in B} \|\bar{v}_y\| = \sup_{y \in B} (A|d_G(y) - \tfrac{1}{2}| + v_\infty(y)).$$

By 1.13, the formula (1.2) holds for any $\psi \in C(B)$. This together with (3.4) implies

$$(3.11) \quad NU = (\tfrac{1}{2}AI + \overline{W})^*,$$

where $(\dots)^*$ denotes the operator adjoint to $(\dots)$.

3.5. REMARK. In §4 we shall be engaged with the Neumann problem in the following formulation: Given $v \in C^*(B)$ find a $\mu \in C^*(B)$ with $NU\mu = v$. By (3.11), this problem reduces to solving the equation

$$(\tfrac{1}{2}AI + \overline{W})^*\mu = v.$$

In connection with this equation it is useful to know the Fredholm radius of $\overline{W}$, i.e., the reciprocal of

$$\omega\overline{W} = \inf_T \|\overline{W} - T\|,$$

where T ranges over all compact operators on $C(B)$ (cf. [RS]). Our main objective in §3 is to express $\omega\overline{W}$ in terms of geometric quantities connected with G and investigate relations between $\omega\overline{W}$ and regularity of B .

3.6. THEOREM. Let I_B denote the set of all isolated points of B and put $E = B - I_B$ if I_B is finite, $E = B$ in the opposite case. Let $V_r = 0$ or

$$V_r = \sup_{y \in E} [A|\tfrac{1}{2} - d_G(y)| + v_r(y)]$$

according as $E = \emptyset$ or not and define

$$V_0 = \lim_{r \rightarrow 0+} V_r.$$

Then $\omega\overline{W} = V_0$.

Proof will be divided into two steps.

Step 1. We first prove that

$$(3.12) \quad \omega\overline{W} \leq V_r$$

for every $r > 0$ satisfying

$$(3.13) \quad H_{m-1}(\hat{B} \cap \{z : |z - y| = r\}) = 0 \quad \text{for all } y,$$

where $\hat{B}$ is the reduced boundary defined in 2.11. If R is the set of all $r > 0$ enjoying (3.13) then $(0, \infty) - R$ is at most countable, because spherical shells with different radii meet each other in a set of H_{m-1} -measure zero and $H_{m-1}(\hat{B}) < \infty$. Hence $V_0 = \inf \{V_r : r \in R\}$ and

$$(3.14) \quad \omega \bar{W} \leq V_0$$

will follow from (3.12). So let us fix $r \in R$. If I_B is finite we assume, as we may, $r < \text{dist}(I_B, E) = \inf \{\text{dist}(z, E) : z \in I_B\}$. Let c_y denote the characteristic function of $B - (\Omega_r(y) \cap E)$ and put

$$W_r f(y) = \int_B c_y f d\bar{v}_y, \quad f \in C(B),$$

where $\bar{v}_y$ is defined by (3.8). Absolute values of all the functions in

$$(3.15) \quad \{W_r f : f \in C(B), \|f\| \leq 1\}$$

are bounded by $\sup_{y \in B} \|\bar{v}_y\| \leq \frac{1}{2}A + V$. If $f \in C(B)$ and x, y are arbitrary points in E with $|x - y| = d \leq \frac{1}{2}r$, then we obtain from 2.12

$$W_r f(x) - W_r f(y) = J_1(f) + J_2(f),$$

where

$$J_1(f) = \int_B f(z) [c_x(z) - c_y(z)] \frac{n(z) \cdot (z - x)}{|z - x|^m} dH_{m-1}(z),$$

$$J_2(f) = \int_B f(z) c_y(z) \left[\frac{z - x}{|z - x|^m} - \frac{z - y}{|z - y|^m} \right] \cdot n(z) dH_{m-1}(z).$$

Denoting

$$\alpha(d) = \sup_z H_{m-1}[(\text{cl } \Omega_{r+d}(z) - \Omega_{r-d}(z)) \cap \hat{B}]$$

we get for $\|f\| \leq 1$

$$|J_1(f)| \leq (\tfrac{1}{2}r)^{1-m} \alpha(d),$$

$$|J_2(f)| \leq (m+1)d(\tfrac{1}{2}r)^{-m}.$$

Since $r \in R$, an easy compactness argument yields

$$\lim_{d \rightarrow 0+} \alpha(d) = 0.$$

We see that all the functions in (3.15) are equicontinuous on E ; noting that $B - E$ is finite we conclude that they are equicontinuous on B as well and the operator

$$W_r : f \rightarrow W_r f$$

is compact. Hence

$$\omega \bar{W} \leq \|\bar{W} - W_r\|.$$

If $f \in C(B)$ then

$$W_\tau f(y) = \overline{W}f(y) \quad \text{for } y \in B - E$$

while (3.9) shows that, for $y \in E$,

$$(\overline{W} - W_\tau)f(y) = \int f d\bar{\nu}_y$$

with the integral extended over $B \cap \Omega_\tau(y)$. Consequently,

$$\|\overline{W} - W_\tau\| = \sup_{y \in E} |\bar{\nu}_y|(\Omega_\tau(y) \cap B) = V_\tau$$

and (3.12) is established.

Step 2. Now we are going to prove the inequality

$$(3.16) \quad \omega \overline{W} \geq V_0$$

which is trivial if $E = \emptyset$. Therefore we assume $E \neq \emptyset$, so that E is infinite. A point $y \in B$ will be termed a discontinuity for a $\mu \in C^*(B)$ if $\mu(\{y\}) \neq 0$. By the Radon theorem, every compact operator on $C(B)$ can be arbitrarily closely approximated by operators of finite rank. If Q is such an operator, sending $f \in C(B)$ into

$$(3.17) \quad Qf = \sum_{k=1}^q g_k \langle f, m_k \rangle,$$

where $g_k \in C(B)$ and $m_k \in C^*(B)$, then every m_k can be arbitrarily closely (in the norm of $C^*(B)$) approximated by $\bar{m}_k \in C^*(B)$ having only a finite number of discontinuities. Defining

$$\bar{Q} \cdots = \sum_{k=1}^q g_k \langle \cdots, \bar{m}_k \rangle$$

we see that the deviation $\|Q - \bar{Q}\|$ can be made as small as we want. It follows from these observations that, in order to prove (3.16), it is sufficient to show that

$$(3.18) \quad \|\overline{W} - Q\| \geq V_0$$

for every Q of the type (3.17), where $m_k \in C^*(B)$ have only a finite number of discontinuities each. Let us fix such a Q and denote by K the (finite) set of all $y \in B$ which represent a discontinuity for some of the measures m_k ($k = 1, \dots, q$). Every m_k splits into m_k^1 having no discontinuities and a finite combination of Dirac measures, to be denoted by m_k^2 . Since y is the only possible discontinuity for $\bar{\nu}_y$, we have for $y \in B - K$

$$\left\| \bar{\nu}_y - \sum_k g_k(y) m_k \right\| = \left\| \bar{\nu}_y - \sum_k g_k(y) m_k^1 \right\| + \left\| \sum_k g_k(y) m_k^2 \right\|,$$

whence

$$\|\overline{W} - Q\| \geq \sup \left\{ \left\| \bar{\nu}_y - \sum_k g_k(y) m_k^1 \right\| : y \in E - K \right\}.$$

Since the operator

$$f \rightarrow \left\langle f, \bar{v}_y - \sum_k g_k(y) m_k^1 \right\rangle$$

sends each $f \in C(B)$ into a continuous function of y we conclude that

$$a_r(y) = \left| \bar{v}_y - \sum_k g_k(y) m_k^1 \right| (\Omega_r(y) \cap B)$$

is a lower semicontinuous function of y for every $r > 0$. Consequently,

$$\begin{aligned} \|\bar{W} - Q\| &\geq \sup \{a_r(y) : y \in E - K\} \\ (3.19) \quad &= \sup \{a_r(z) : z \in E - (I_B \cap K)\}. \end{aligned}$$

Consider now an arbitrary $y \in E \cap I_B \cap K$ and note that $E \cap I_B \neq \emptyset$ implies

$$\emptyset \neq E \cap (I_B - K) \subset E - (I_B \cap K).$$

If

$$r < \text{dist}(I_B \cap K, E - (I_B \cap K)),$$

then

$$\begin{aligned} (3.20) \quad \left| \sum_k g_k(y) m_k^1 \right| (\Omega_r(y) \cap B) &= \left| \sum_k g_k(y) m_k^1 \right| (\Omega_r(y) \cap I_B) = 0, \\ a_r(y) &= |\bar{v}_y| (\Omega_r(y) \cap I_B) = \frac{1}{2}A. \end{aligned}$$

On the other hand, we have for any $z \in I_B$

$$\frac{1}{2}A \leq |A(\frac{1}{2} - d_G(z)\delta_z)| (\Omega_r(z) \cap B) + \left| \nu_z - \sum_k g_k(z) m_k^1 \right| (\Omega_r(z) \cap B) = a_r(z),$$

because $\nu_z - \sum_k g_k(z) m_k^1$ has no discontinuities. Combining this with (3.20) we get

$$a_r(y) \leq \sup \{a_r(z) : z \in E \cap (I_B - K)\} \leq \sup \{a_r(z) : z \in E - (I_B \cap K)\}.$$

We have thus for small $r > 0$

$$(3.21) \quad \sup \{a_r(z) : z \in E - (I_B \cap K)\} = \sup \{a_r(y) : y \in E\}.$$

Note that

$$V_r = \sup_{y \in E} |\bar{v}_y| (\Omega_r(y) \cap B).$$

If $M = \max \{|g_k(x)| : x \in B, 1 \leq k \leq q\}$, then

$$(3.22) \quad \sup \{a_r(y) : y \in E\} \geq V_r - M \sum_k \sup_{y \in E} |m_k^1| (\Omega_r(y) \cap B).$$

Since m_k^1 ($k = 1, \dots, q$) have no discontinuities,

$$\lim_{r \rightarrow 0+} \sup_{y \in B} |m_k^1|(\Omega_r(y) \cap B) = 0.$$

Making $r \rightarrow 0+$ in (3.22) and using (3.21), (3.19) we arrive at (3.18).

REMARK. The basic idea of the above proof goes back to J. Radon (cf. [RS]).

3.7. LEMMA. *Let us define $\hat{B}$ as in 2.11 and put*

$$B^* = B \cap \{y : |d_G(y) - \frac{1}{2}| < \frac{1}{2}\}.$$

Then $\hat{B}$ is dense in B^ (moreover, every ball of center in B^* meets $\hat{B}$ in a set of positive H_{m-1} -measure) and*

$$H_{m-1}(B^* - \hat{B}) = 0.$$

Proof. If $y \in B^*$ then there is an $\varepsilon > 0$ such that

$$H_m(\Omega_r(y) \cap G) > \varepsilon H_m(\Omega_r(y)),$$

$$H_m(\Omega_r(y) \cap C) > \varepsilon H_m(\Omega_r(y))$$

for $0 < r < \varepsilon$. By the relative isoperimetric inequality for sets with finite perimeter (cf. Theorem (4.3) in [MI]; general isoperimetric inequalities for currents may be found in [FF, §6]) we conclude that

$$H_{m-1}(\Omega_r(y) \cap \hat{B}) \geq \alpha r^{m-1}, \quad 0 < r < \varepsilon,$$

where $\alpha > 0$ does not depend on r . Hence it follows by [F2, §3] that

$$H_{m-1}(B^* - \hat{B}) = 0.$$

3.8. Notation. For $z \in R^m$, $r > 0$ and $\theta \in \Gamma$ we put

$$\Omega_r(z, \theta) = \Omega_r(z) \cap \{x : (x - z) \cdot \theta > 0\}.$$

We denote by $a(\theta, \eta) = \arccos(\theta \cdot \eta)$ the nonoriented angle enclosed by $\theta, \eta \in \Gamma$. It is easily seen that

$$(3.23) \quad \frac{a(\theta, \eta)}{2\pi} = \frac{H_m(\Omega_r(z, \theta) \cap \Omega_r(z, -\eta))}{H_m(\Omega_r(z))}.$$

The symbol n will always have the meaning described in 2.11. The symmetric difference of $P, Q \subset R^m$ will be denoted by $P \dot{-} Q$.

3.9. LEMMA. *Let $z \in \hat{B}$, $\theta = n(z)$. Then*

$$H_m(\Omega_r(z, \theta) \cap \text{int } C) + H_m(\Omega_r(z, -\theta) \cap G) \leq H_m(\Omega_r(z)) \frac{v_r(z)}{A}.$$

Proof. Let

$$\gamma_1 < \frac{H_m(\Omega_r(z, \theta) \cap \text{int } C)}{H_m(\Omega_r(z))}, \quad \gamma_2 < \frac{H_m(\Omega_r(z, -\theta) \cap G)}{H_m(\Omega_r(z))}.$$

Put $\Gamma_+ = \Gamma \cap \{\eta : \eta \cdot \theta > 0\}$, $\Gamma_- = \Gamma \cap \{\eta : \eta \cdot \theta < 0\}$, $S(\rho) = \{x : |x - z| = \rho\}$ and define K_z, L_z as in 2.6. There are $\rho_1, \rho_2 \in (0, r)$ such that

$$(3.24) \quad H_{m-1}(S(\rho_1) \cap \Omega_r(z, \theta) \cap \text{int } C) > \gamma_1 H_{m-1}(S(\rho_1)),$$

$$(3.25) \quad H_{m-1}(S(\rho_2) \cap \Omega_r(z, -\theta) \cap G) > \gamma_2 H_{m-1}(S(\rho_2)).$$

By virtue of 2.7

$$H_{m-1}(L_z \cap \Gamma_+) = \frac{1}{2}A \lim_{\rho \rightarrow 0+} \frac{H_m(\Omega_\rho(z, \theta) \cap C)}{H_m(\Omega_\rho(z, \theta))} = 0,$$

$$H_{m-1}(L_z \cap \Gamma_-) = \frac{1}{2}A \lim_{\rho \rightarrow 0+} \frac{H_m(\Omega_\rho(z, -\theta) \cap C)}{H_m(\Omega_\rho(z, -\theta))} = \frac{1}{2}A.$$

We see that L_z is equivalent (H_{m-1}) with Γ_- and K_z is equivalent (H_{m-1}) with Γ_+ . If $\eta \in L_z$ and $\{z + \rho\eta : 0 < \rho < r\} \cap G \neq \emptyset$ then, with the notation from 1.6, $n_r(\eta, z) \geq 1$. Employing (3.25) we obtain

$$\int_{L_z} n_r(\eta, z) dH_{m-1}(\eta) > \gamma_2 A.$$

Similarly, (3.24) implies

$$\int_{K_z} n_r(\eta, z) dH_{m-1}(\eta) > \gamma_1 A,$$

so that

$$v_r(z) = \int_{\Gamma} n_r(\eta, z) dH_{m-1}(\eta) > (\gamma_1 + \gamma_2)A.$$

3.10. LEMMA. Let $N \in \Gamma$, $y \in R^m$, $r > 0$ and suppose that

$$(3.26) \quad \sup_{z \in B} v_r(z) \leq u_0 A,$$

$$(3.27) \quad H_m(\Omega_r(y, N) \cap C) \leq u_1 H_m(\Omega_r(y)),$$

$$(3.28) \quad H_m(\Omega_r(y, -N) \cap \text{cl } G) \leq u_2 H_m(\Omega_r(y)).$$

If $s = u_0 + u_1 + u_2 < \frac{1}{2}$, then for every $\gamma > s$ there is a $\delta > 0$ (depending on $(\gamma - s)r$ only) such that

$$a(n(z), N) \leq \pi\gamma \quad \text{for } z \in \hat{B} \cap \Omega_\delta(y).$$

Proof. Let $\gamma = s + 2\epsilon$, $\epsilon > 0$, and consider a $\theta \in \Gamma$ with $a(N, \theta) > \gamma\pi$. We have by (3.23)

$$H_m(\Omega_r(y, -N) \cap \Omega_r(y, \theta)) > \frac{1}{2}\gamma H_m(\Omega_r(y)),$$

$$H_m(\Omega_r(y, N) \cap \Omega_r(y, -\theta)) > \frac{1}{2}\gamma H_m(\Omega_r(y)).$$

Let us fix $\delta > 0$ small enough to secure

$$H_m(\Omega_r(z, \eta) \dot{-} \Omega_r(y, \eta)) < \varepsilon H_m(\Omega_r(y))$$

for $|z - y| < \delta$ and any $\eta \in \Gamma$. We have then for $z \in \Omega_\delta(y)$

$$H_m(\Omega_r(y, -N) \cap \Omega_r(z, \theta)) > \frac{1}{2}s H_m(\Omega_r(z)),$$

$$H_m(\Omega_r(y, N) \cap \Omega_r(z, -\theta)) > \frac{1}{2}s H_m(\Omega_r(z)),$$

whence we obtain on account of (3.27), (3.28)

$$(3.29) \quad \begin{aligned} H_m(\Omega_r(z, \theta) \cap \text{int } C) &\geq H_m(\Omega_r(z, \theta) \cap \Omega_r(y, -N)) - H_m(\Omega_r(y, -N) \cap \text{cl } G) \\ &> (\frac{1}{2}s - u_2) H_m(\Omega_r(z)), \end{aligned}$$

$$(3.30) \quad \begin{aligned} H_m(\Omega_r(z, -\theta) \cap G) &\geq H_m(\Omega_r(z, -\theta) \cap \Omega_r(y, N)) \\ - H_m(\Omega_r(y, N) \cap C) &> (\frac{1}{2}s - u_1) H_m(\Omega_r(z)). \end{aligned}$$

Suppose now that $z \in \hat{B}$ and $\theta = n(z)$. Employing (3.29), (3.30) and Lemma 3.9 we arrive at $v_r(z) > u_0 A$, which violates (3.26).

3.11. Notation. Let P_N stand for the orthogonal projection of R^m onto $R^m \cap \{x : x \cdot N = 0\}$. With every $\alpha \in (0, \frac{1}{2})$ we associate $B(\alpha) \subset B$ as follows. We let $y \in B(\alpha)$ if for every $\gamma \in (\alpha, \frac{1}{2})$ there is a neighborhood Q of y in B and an $N \in \Gamma$ such that $|P_N(x) - P_N(z)| \geq |x - z| \cos \pi \gamma$ whenever $x, z \in Q$. By Theorem 5.1 in [MI] we get the following corollary of Lemma 3.10:

3.12. COROLLARY. *If (3.26), (3.27), (3.28) hold and $s = u_0 + u_1 + u_2 < \frac{1}{2}$, then $y \in B(s)$; moreover, for every $\gamma \in (s, \frac{1}{2})$ there is a $\delta > 0$ (depending on $r(\gamma - s)$ only) such that $B \cap \Omega_\delta(y) \subset B(\gamma)$.*

3.13. THEOREM. *If $V_0 < \frac{1}{2}A$ then I_B is finite and*

$$(3.31) \quad H_{m-1}(B - B(V_0/A)) = 0.$$

If $V_0 < \frac{1}{4}A$ then $B = B(2V_0/A)$.

Proof. Let $V_0 < \frac{1}{2}A$. Then I_B must be finite, $B - I_B = E \subset B^*$ and, by 3.7, $H_{m-1}(B - \hat{B}) = 0$. To prove (3.31) it is therefore sufficient to show that

$$\hat{B} \subset B\left(3\varepsilon + \frac{V_0}{A}\right)$$

for every small $\varepsilon > 0$. Fix $y \in \hat{B}$, $N = n(y)$ and $\varepsilon > 0$, $3\varepsilon + V_0/A < \frac{1}{2}$. We have then for sufficiently small $r > 0$

$$(3.32) \quad \begin{aligned} H_m(\Omega_r(y, N) \cap C) &\leq \varepsilon H_m(\Omega_r(y)), \\ H_m(\Omega_r(y, -N) \cap \text{cl } G) &= H_m(\Omega_r(y, -N) \cap G) \\ &\leq \varepsilon H_m(\Omega_r(y)), \\ V_r &< V_0 + \varepsilon A. \end{aligned}$$

Employing 3.12 we get $y \in B(3\varepsilon + V_0/A)$. Suppose now that $\alpha = V_0/A < \frac{1}{4}$ and fix an $r > 0$ with (3.32). By 3.9 we have for all $y \in \hat{B}$

$$H_m(\Omega_r(y, n(y)) \cap C) + H_m(\Omega_r(y, -n(y)) \cap \text{cl } G) < (\alpha + \varepsilon)H_m(\Omega_r(y)).$$

By 3.12 there is a $\delta > 0$ independent of y such that $\Omega_\delta(y) \cap B \subset B(3\varepsilon + 2\alpha)$ for every $y \in \hat{B}$. It remains to note that $\hat{B}$ is dense in E by 3.7.

3.14. COROLLARY. If $V_0 < \frac{1}{4}A$ then

$$\lim_{r \rightarrow 0+} \sup \{ \rho^{1-m} H_{m-1}(\Omega_\rho(y) \cap B) : y \in B, 0 < \rho < r \} \leq b_{m-1} \sec(2V_0\pi/A),$$

where b_{m-1} denotes the volume of the unit ball in R^{m-1} .

3.15. THEOREM. Let $V_0 = 0$ (which means that $\bar{W}$ is compact). Then I_B is finite and $E = B - I_B$ is a surface of class C^1 .

Proof. For every $\varepsilon > 0$, $\varepsilon < \frac{1}{4}$ there is an $r > 0$ such that $V_r < \varepsilon A$. By 3.9 (note also that $H_m(B) = 0$)

$$H_m(\Omega_r(y, n(y)) \cap C) + H_m(\Omega_r(y, -n(y)) \cap \text{cl } G) < \varepsilon H_m(\Omega_r(y))$$

for all $y \in \hat{B}$. Employing 3.10 we get a $\delta > 0$ depending on $r\varepsilon$ only such that, for every couple of points $y, z \in \hat{B}$, $a(n(z), n(y)) \leq 3\varepsilon\pi$ whenever $|y - z| < \delta$. We see that n is uniformly continuous on $\hat{B}$. By 3.7, n extends to a continuous function N on $E = \text{cl } \hat{B}$ and, for every $y \in E$,

$$N(y) = \lim_{r \rightarrow 0+} \frac{\int N(z) dH_{m-1}(z)}{H_{m-1}(\Omega_r(y) \cap E)}$$

with the integral extended over $\Omega_r(y) \cap E$. Hence it follows by [DG3, Theorem III] (see also definition of the reduced boundary presented in [DG3, p. 10]) that E is a surface of class C^1 .

REMARK. The main results of this paper (such as Theorem 1.13 or Theorem 3.6) are expressed in terms of the quantity $v_r(y)$. In the definition of $v_r(y)$ one considers all half-lines issuing at y , i.e., orthogonal trajectories of the level surfaces of the Green function with a fixed pole at y . This suggests the possibility of generalizing these results to the case of a Green space in the sense of [BC].

4. Boundary value problems.

Notation. We shall keep the notation and assumptions introduced in §3. Besides that we always assume here that $m > 2$ (see Remark 4.10 below dealing with $m = 2$). We shall start with investigation of solutions of the equations

$$(4.1) \quad (\tfrac{1}{2}AI + \bar{W})f = 0 \quad \text{over } C(B),$$

$$(4.2) \quad (\tfrac{1}{2}AI + \bar{W})^*\mu = 0 \quad \text{over } C^*(B).$$

$C_0(B)$ will denote the class of all $f \in C(B)$ satisfying (4.1) and $C^*(B)$ will stand for

the set of all $\mu \in C^*(B)$ satisfying (4.2). We agree to use M as a generic notation for a Borel set. If μ is a signed Borel measure in R^m and $R \subset R^m$ is a fixed Borel set, we define $\mu \cap R$ by

$$\mu \cap R(M) = \mu(M \cap R), \quad M \subset R^m.$$

Recalling the definition of $\bar{\nu}_y$ presented in (3.8) we obtain from (3.9) that, for every $\mu \in C^*(B)$,

$$(4.3) \quad \bar{W}^*\mu(M) = \int_B \bar{\nu}_y(M) d\mu(y), \quad M \subset B.$$

It follows from (3.10) that

$$(4.4) \quad \|W^*\| \leq \frac{1}{2}A + V,$$

where $V = V^c$ has been defined in 2.13.

4.1. LEMMA. *If $\mu \in C_0^*(B)$ then $|\mu|(I_B) = 0$ (see 3.6 for notation).*

Proof. Let $\mu \in C_0^*(B)$, $z \in I_B$ and denote by f the characteristic function of $\{z\}$. We have by 3.4, 1.13, and (3.3)

$$0 = NU\mu(\{z\}) = \int_B (Ad_G(y)f(y) + W^cf(y)) d\mu(y).$$

It follows from (2.12) that $W^cf = 0$, so that $Ad_G(y)f(y) + W^cf(y) = Af(y)$ for all $y \in B$. Hence $\mu(\{z\}) = 0$.

REMARK. A refinement of the preceding argument may be used to show that, for every $\mu \in C_0^*(B)$, $\mu \cap M$ is absolutely continuous with respect to $H_{m-1} \cap \hat{B}$ provided $d_G(y) > 0$ for all $y \in M$.

As it follows from 4.1, $C_0^*(B)$ contains only trivial measure in case $B = I_B$. In what follows we always exclude the trivial case of a finite B .

4.2. LEMMA. *Fix $z \in B$, $\mu \in C^*(B)$ and put for $t > 0$*

$$(4.5) \quad R_t = B \cap \Omega_t(z),$$

$$(4.6) \quad \alpha(t) = H_{m-1}(R_t \cap \hat{B}),$$

$$(4.7) \quad \beta(t) = |\mu|(R_t).$$

Let $0 < \rho < \delta < \Delta$ and suppose that

$$\mu \cap (R_\Delta - R_\delta) = \mu.$$

Then

$$(4.8) \quad |\bar{W}^*\mu|(R_\rho) \leq \frac{\alpha(\rho)\beta(\Delta)}{(\Delta - \rho)^{m-1}} + (m-1)\alpha(\rho) \int_\delta^\Delta \frac{\beta(t) dt}{(t - \rho)^m}.$$

Proof. Let g denote the characteristic function of $\hat{B} \cap R_\rho$. By 2.12 we obtain for $y \in R_\Delta - R_\delta$ and $M \subset R_\rho$

$$|\bar{\nu}_y(M)| = |\nu_y(M)| \leq \int_M g(x) |y-x|^{1-m} dH_{m-1}(x),$$

whence it follows easily by (4.3)

$$(4.9) \quad |\bar{W}^* \mu|(R_\rho) \leq \iint_{B \times B} g(x) |y-x|^{1-m} dH_{m-1}(x) d|\mu|(y).$$

Since

$$\begin{aligned} \int_B |y-x|^{1-m} d|\mu|(y) &\leq \int_\delta^\Delta (t-|x|)^{1-m} d\beta(t) \\ &\leq \frac{\beta(\Delta)}{(\Delta-|x|)^{m-1}} + (m-1) \int_\delta^\Delta \frac{\beta(t) dt}{(t-|x|)^m}, \end{aligned}$$

(4.9) implies (4.8).

4.3. LEMMA. Fix $z \in B$, $r > 0$ and put, with the notation from 4.2,

$$R = R_r (= \Omega_r(z) \cap B),$$

$$V(R) = \sup \{ |\bar{\nu}_y|(R) : y \in R \},$$

$$Q(R) = \sup \{ \rho^{1-m} \alpha(\rho) : 0 < \rho < r \},$$

$$K(R) = \inf \left\{ V(R)k^{m-2} + Q(R) \left[\left(\frac{k}{k-1} \right)^{m-1} - 1 \right] : k > 1 \right\}.$$

Define

$$\bar{W}_R^* \mu = (\bar{W}^* \mu) \cap R, \quad \mu \in C^*(B).$$

Let C_R^* denote the set of all $\mu \in C^*(B)$ enjoying

$$J(\mu) = \int_0^r \rho^{1-m} |\mu|(R_\rho) d\rho < \infty$$

and

$$(4.10) \quad |\mu|(B-R) = 0$$

and put

$$\|\mu\|_R = \frac{1}{m-2} r^{2-m} \|\mu\| + J(\mu), \quad \mu \in C_R^*.$$

Then $\mu \in C_R^*$ implies $\bar{W}_R^* \mu \in C_R^*$ and

$$\|\bar{W}_R^* \mu\|_R \leq K(R) \|\mu\|_R.$$

Proof. Fix $\mu \in C_R^*$ and $k > 1$. We have with the notation from (4.7)

$$(4.11) \quad J(\mu) = \int_0^r \rho^{1-m} \beta(\rho) d\rho.$$

Let now $0 < \rho < r/k$ and define

$$\mu_\rho = \mu \cap R_{k\rho}, \quad \mu^\rho = \mu - \mu_\rho.$$

In view of (4.3)

$$(4.12) \quad \|\overline{W}_R^* \mu_\rho\| \leq V(R)\beta(k\rho).$$

Employing 4.2 we obtain

$$(4.13) \quad |\overline{W}_R^* \mu^\rho|(R_\rho) \leq \frac{\alpha(\rho)\beta(r)}{(r-\rho)^{m-1}} + (m-1)\alpha(\rho) \int_{k\rho}^r \frac{\beta(t) dt}{(t-\rho)^m}.$$

On account of (4.12), (4.13) we get for $0 < \rho < r/k$

$$\rho^{1-m} |\overline{W}_R^* \mu|(R_\rho) \leq V(R)\rho^{1-m}\beta(k\rho) + Q(R) \frac{\beta(r)}{(r-\rho)^{m-1}} + (m-1)Q(R) \int_{k\rho}^r \frac{\beta(t) dt}{(t-\rho)^m},$$

while, by (4.3),

$$\rho^{1-m} |\overline{W}_R^* \mu|(R_\rho) \leq V(R)\rho^{1-m}\beta(r) \quad \text{for } r/k \leq \rho < r.$$

Using (4.11) we obtain after simple calculation

$$\begin{aligned} J(\overline{W}_R^* \mu) &= \int_0^{r/k} \rho^{1-m} |\overline{W}_R^* \mu|(R_\rho) d\rho + \int_{r/k}^r \rho^{1-m} |\overline{W}_R^* \mu|(R_\rho) d\rho \\ &\leq \frac{\beta(r)}{m-2} r^{2-m} \left(V(R)(k^{m-2}-1) + Q(R) \left[\left(\frac{k}{k-1} \right)^{m-2} - 1 \right] \right) \\ &\quad + J(\mu) \left(V(R)k^{m-2} + Q(R) \left[\left(\frac{k}{k-1} \right)^{m-1} - 1 \right] \right). \end{aligned}$$

Since, by virtue of (4.3),

$$\|\overline{W}_R^* \mu\| \leq V(R)\beta(r),$$

we get finally

$$\begin{aligned} \|\overline{W}_R^* \mu\|_R &= \frac{1}{m-2} r^{2-m} \|\overline{W}_R^* \mu\| + J(\overline{W}_R^* \mu) \\ &\leq \|\mu\|_R \left(V(R)k^{m-2} + Q(R) \left[\left(\frac{k}{k-1} \right)^{m-1} - 1 \right] \right). \end{aligned}$$

4.4. Notation. Let

$$Q_r = \sup \{ \rho^{1-m} H_{m-1}(\Omega_\rho(z) \cap \hat{B}) : z \in B, 0 < \rho < r \}, \quad r > 0,$$

$$Q_0 = \lim_{r \rightarrow 0+} Q_r.$$

Further define V_0 as in 3.6 and put

$$(4.14) \quad K_0 = \inf \left\{ V_0 k^{m-2} + Q_0 \left[\left(\frac{k}{k-1} \right)^{m-1} - 1 \right] : k > 1 \right\}.$$

In what follows we shall always assume that

$$(4.15) \quad K_0 < \frac{1}{2}A.$$

4.5. REMARK. The inequality (4.15) implies

$$(4.16) \quad V_0 < \frac{1}{4}A.$$

Indeed, since B is infinite and $V_0 < \frac{1}{2}A$, (3.7) secures $H_{m-1}(\hat{B}) > 0$. It is known from [DG2], [F3] that for (H_{m-1}) almost all $y \in \hat{B}$

$$\lim_{\rho \rightarrow 0+} \rho^{1-m} H_{m-1}(\Omega_\rho(y) \cap \hat{B}) = b_{m-1},$$

where b_{m-1} denotes the volume of the unit ball in R^{m-1} . Hence $Q_0 \geq b_{m-1}$ and k minimizing

$$V_0 k^{m-2} + Q_0 \left[\left(\frac{k}{k-1} \right)^{m-1} - 1 \right]$$

must satisfy

$$\frac{1}{2}A > Q_0 \left[\left(\frac{k}{k-1} \right)^{m-1} - 1 \right] \geq b_{m-1} \frac{m-1}{k-1},$$

which guarantees $k^{m-2} > 2$.

On the other hand, if (4.16) holds, then 3.14 provides an estimate for Q_0 in terms of V_0 . Clearly, (4.15) is fulfilled whenever V_0 is sufficiently small.

In view of (4.16) and 3.6, the Fredholm theory applies to the pair of adjoint equations

$$(\frac{1}{2}AI + \overline{W})f = g,$$

$$(\frac{1}{2}AI + \overline{W})^* \mu = v.$$

4.6. LEMMA. If $\mu \in C_0^*(B)$ then $U|\mu|$ (see the introduction for notation) is bounded on B .

Proof. Define V_r and E as in 3.6 and fix $r > 0$ and $k > 1$ such that

$$K = V_{2r} k^{m-2} + Q_r \left[\left(\frac{k}{k-1} \right)^{m-1} - 1 \right] < \frac{1}{2}A,$$

$$r < \text{dist}(E, B-E).$$

Fix an arbitrary $z \in E$ and define $R = \Omega_r(z) \cap B$. We have then with the notation from 4.3

$$V(R) \leq V_{2r}, \quad Q(R) \leq Q_r,$$

$$(4.17) \quad K(R) \leq K < \frac{1}{2}A.$$

Let $\mu \in C_0^*(B)$, $\|\mu\| \leq 1$ and put

$$\mu_0 = \mu \cap R, \quad \mu^0 = \mu - \mu_0.$$

In view of (4.2)

$$(4.18) \quad \frac{1}{2}A\mu_0 + \overline{W}^*\mu_0 = -\frac{1}{2}A\mu^0 - \overline{W}^*\mu^0.$$

Restricting all measures occurring in (4.18) to R we obtain

$$(4.19) \quad (I + 2A^{-1}\overline{W}_R^*)\mu_0 = -2A^{-1}\overline{W}_R^*\mu^0$$

where, of course, I is the identity operator. Employing 4.2 with $\delta=r$ and $\Delta=r + \text{diam } B$ we obtain easily for $0 < \rho \leq r/2$

$$\rho^{1-m}|\overline{W}^*\mu^0|(R_\rho) \leq Q_r 2^m r^{1-m} \|\mu^0\| \leq Q_r 2^m r^{1-m}.$$

On the other hand, we have for $\rho > r/2$

$$\rho^{1-m}|\overline{W}^*\mu^0|(R_\rho) \leq 2^{m-1}r^{1-m} \|\overline{W}^*\| \cdot \|\mu^0\| \leq 2^{m-1}r^{1-m}(\frac{1}{2}A + V),$$

so that

$$J(\overline{W}_R^*\mu^0) \leq 2^{m-2}r^{2-m}(2Q_r + \frac{1}{2}A + V).$$

Since

$$\|\overline{W}_R^*\mu^0\| \leq \|\overline{W}^*\mu^0\| \leq \frac{1}{2}A + V$$

we arrive at

$$\|\overline{W}_R^*\mu^0\|_R \leq \gamma_r,$$

where

$$\gamma_r = \frac{1}{m-2} r^{2-m}(\frac{1}{2}A + V) + 2^{m-2}r^{2-m}(2Q_r + \frac{1}{2}A + V).$$

We see that $\overline{W}_R^*\mu^0 \in C_R^*$. It is easily seen that C_R^* , equipped with the norm $\|\cdots\|_R$, is a Banach space. In view of (4.3) and (4.17)

$$\|\overline{W}_R^*\|_R \leq K < \frac{1}{2}A.$$

Hence we conclude by virtue of (4.19) that $\mu_0 \in C_R^*$ and

$$\|\mu_0\|_R \leq \left(1 - \frac{2K}{A}\right)^{-1} 2A^{-1}\gamma_r = a_r.$$

Since a_r is independent of $z \in E$, we have, in particular,

$$\sup_{z \in E} \int_0^r \rho^{1-m} |\mu|(\Omega_\rho(z) \cap B) d\rho < \infty,$$

whence it follows easily

$$\sup_{z \in E} \int_0^\infty \rho^{1-m} |\mu|(\Omega_\rho(z) \cap B) d\rho < \infty.$$

Noting that

$$\begin{aligned} U|\mu|(z) &= \frac{1}{m-2} \int_B |x-z|^{2-m} d|\mu|(x) \\ &= \frac{1}{m-2} \int_0^\infty |\mu|(B \cap \{x : |x-z|^{2-m} > t\}) dt \\ &= \int_0^\infty \rho^{1-m} |\mu|(\Omega_\rho(z) \cap B) d\rho \end{aligned}$$

we see that $U|\mu|$ is bounded on E . Since, by 4.1, $\text{spt } \mu \subset E$ and $B-E$ has a positive distance from E , $U|\mu|$ is bounded on B as well.

4.7. Notation. It follows easily from (4.16) and 3.13 that G has only a finite number of components; their closures are mutually disjoint. We shall denote by $q(0 \leq q < \infty)$ the number of bounded components of G . G_0 will stand for the unbounded component of G (if any); the bounded components of G will be denoted by $G_1, \dots, G_q$.

Employing 4.6 we obtain by standard reasoning the following.

4.8. LEMMA. *The dimension of $C_0^*(B)$ does not exceed q .*

Proof. Let $\mu \in C_0^*(B)$. By 4.6, $U|\mu|$ is bounded on B . Hence it follows that μ has finite energy [B, p. 122] and

$$\int_{\mathbb{R}^m} |\text{grad } U\mu(x)|^2 dx = A \int_B U\mu(y) d\mu(y) < \infty$$

(see [B, pp. 131, 132]). In particular, there are $\phi_k \in D$ such that

$$\int_{\mathbb{R}^m} |\text{grad } \phi_k(x) - \text{grad } U\mu(x)|^2 dx \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

(4.2) means that $NU\mu = 0$ (see 3.4), so that

$$\int_G \text{grad } \phi_k(x) \cdot \text{grad } U\mu(x) dx = 0$$

for each k ; making $k \rightarrow \infty$ we obtain

$$\int_G |\text{grad } U\mu(x)|^2 dx = 0.$$

We see that $U\mu$ is constant on each G_j and vanishes on G_0 . Next we prove the following assertion:

(a) *If $U\mu = 0$ on G then $\mu = 0$.*

Indeed, let $\mu = \mu_1 - \mu_2$ be the Jordan decomposition of μ and assume that $U\mu_1$ and $U\mu_2$ coincide on G . Since G has a positive m -dimensional density at any $z \in B$, every fine neighborhood of z (in the Cartan topology) meets G (compare

[B, p. 78, paragraph 2, §3 and p. 84, paragraph 6]) and we conclude from the Cartan Theorem [B, p. 86; see also p. 84] that $U_{\mu_1}(z) = U_{\mu_2}(z)$. Since U_{μ_1} and U_{μ_2} coincide on B , they must coincide on R^m , by the domination principle [B, p. 123]. We have thus $U\mu = 0$ on R^m , whence $\mu = 0$ [B, p. 122].

If $q = 0$ then (a) completes the proof of 4.8. Assume now $q > 0$. With every $\mu \in C_0^*(B)$ we may associate the q -tuple $c(\mu) = [c_1(\mu), \dots, c_q(\mu)]$, where $c_j(\mu)$ is the value taken on by $U\mu$ in G_j . The map

$$c: \mu \rightarrow c(\mu)$$

is an injection of $C_0^*(B)$ into R^q . Indeed, $c(\mu) = 0$ means that $U\mu = 0$ on G and (a) implies $\mu = 0$.

4.9. PROPOSITION. *Let f_j denote the characteristic function of $\text{fr } G_j$ ($1 \leq j \leq q$). Then $\{f_1, \dots, f_q\}$ is a basis in $C_0(B)$.*

Proof. Let us fix $j \in \langle 1, q \rangle$ and put $H = R^m - G_j$. Employing 2.12 and 2.6 we obtain for any $z \in R^m - \text{cl } G \subset \text{int } H$

$$W^c f_j(z) = \nu_z^H(\text{fr } G_j) = 0,$$

whence it follows by (3.6)

$$(\tfrac{1}{2}AI + \overline{W})f_j = 0,$$

so that $f_j \in C_0(B)$. Since the dimension of $C_0(B)$ coincides with the dimension of $C_0^*(B)$ which is known to be $\leq q$ and $f_1, \dots, f_q$ are linearly independent, the proof is complete.

4.10. REMARK. Combining the above proposition and Fredholm's theorems one obtains Theorems 4.11–4.13 below.

If $m = 2$ then 4.9 holds under more general assumptions on B . It is sufficient to require that E (see 3.6) consists of mutually disjoint simple closed curves and $V_0 < \tfrac{1}{2}A$ (compare [K3], where further references may be found).

4.11. THEOREM. *Let $\nu \in C^*(B)$. Then $\nu = NU\mu$ for some $\mu \in C^*(B)$ if and only if*

$$\nu(\text{fr } G_j) = 0, \quad j = 1, \dots, q.$$

Proof. This follows at once from 4.9 and the Fredholm Theorem.

4.12. THEOREM. *Let $\{f_1, \dots, f_q\}$ be a basis in $C_0(B)$. Given $g \in C(B)$ there are $f \in C(B)$ and constants α_j ($j = 1, \dots, q$) such that, for every $y \in B$, $Wf(x)$ tends to*

$$g(y) - \sum_{j=1}^q \alpha_j f_j(y)$$

as $x \rightarrow y$, $x \in \text{int } C$. The constants α_j are uniquely determined and f is determined modulo $C_0(B)$.

Proof. Let $\{\mu_1, \dots, \mu_q\}$ and $\{f_1, \dots, f_q\}$ be dual bases in $C_0^*(B)$ and $C_0(B)$, respectively. Given $g \in C(B)$ we can find α_k so that

$$\left\langle g - \sum_{k=1}^q \alpha_k f_k, \mu_j \right\rangle = 0$$

for all j ; clearly, $\alpha_k = \langle g, \mu_k \rangle$. Then

$$\left\langle g - \sum_{k=1}^q \alpha_k f_k, C_0^*(B) \right\rangle = 0$$

and the Fredholm Theorem yields an $f \in C(B)$ such that

$$(\frac{1}{2}AI + \overline{W})f = g - \sum_{k=1}^q \alpha_k f_k.$$

The rest follows from (3.6).

Standard reasoning yields also the following.

4.13. THEOREM. Fix $x_j \in G_j$ ($j=1, \dots, q$). Given $g \in C(B)$ there are $f \in C(B)$ (determined modulo $C_0(B)$) and uniquely determined constants a_j such that, for every $y \in B$,

$$Wf(x) + \sum_{j=1}^q a_j |x - x_j|^{2-m}$$

tends to $g(y)$ as $x \rightarrow y$, $x \in \text{int } C$.

Proof. Define g_k by

$$g_k(x) = \frac{1}{m-2} |x - x_k|^{2-m}.$$

Then $\langle g_k, \mu \rangle = U\mu(x_k)$ for every $\mu \in C^*(B)$. It follows from (3.6) that

$$Wf + \sum_{j=1}^q \alpha_j g_j \quad (f \in C(B), \alpha_j \in R^1)$$

represents a solution of the Dirichlet problem for C and the boundary condition g if and only if

$$(4.20) \quad (\frac{1}{2}AI + \overline{W})f = g - \sum_{j=1}^q \alpha_j g_j$$

on B . For the existence of an $f \in C(B)$ satisfying (4.20) it is necessary and sufficient that

$$\left\langle g - \sum_{j=1}^q \alpha_j g_j, C_0^*(B) \right\rangle = 0,$$

i.e.,

$$(4.21) \quad \sum_{j=1}^q \alpha_j U\mu(x_j) = \langle g, \mu \rangle, \quad \mu \in C_0^*(B).$$

We know from the proof of 4.8 (note also that $C_0^*(B)$ has dimension q) that

$$\mu \rightarrow [U\mu(x_1), \dots, U\mu(x_q)]$$

is an isomorphism of $C_0^*(B)$ onto R^q . Consequently, (4.21) determines α_j uniquely. The rest is obvious.

REMARK. Results related to some of those proved in the present paper were announced without proofs in [K1] (for the plane), [BMS] and [MS] (for a domain bounded by a simple closed surface in 3-space), [K2] (for a domain bounded by a hyper-surface in m -space) and in Abstract 630-197, (Theorem 1.13), Notices Amer. Math. Soc. **13** (1966).

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